PDE-Based Modeling and Non-collocated Feedback Control of Electrosurgical-Probe/Tissue Interaction

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***Abstract*— The first control-oriented model of the interaction of an electrosurgical probe with organic tissue, based on a 1-D PDE with a moving boundary, is introduced. To attain the desired electrosurgically-induced tissue changes using this model, a non-collocated output feedback moving boundary control law is proposed. The latter is realized through a novel non-collocated pointwise temperature-based state observer for the two-phase Stefan problem. Simulation demonstrates that the controller proposed meets the performance objective. The controller implementation is also discussed.**

I. INTRODUCTION

Over the past decades, the advantages of robot-assisted surgery over conventional laparoscopy have become increasingly apparent [1]. As the robotic surgery performance demands grow, so does the need for a deeper understanding of the physical phenomena governing both the tools and the tissue treated, as well as the control laws for the precise attainment of the surgical objectives.

Electrosurgery relies on the use of high power density radio frequency currents to actively heat organic tissue, allowing it to be denatured, coagulated, desiccated, fulgurated, or incised [2]. One of the key advantages of electrosurgery is its capability of simultaneous cutting and coagulation, providing blood stoppage for complex surgical tasks. Since this technique allows for precise ablation of tissue with very little collateral damage ($\sim$100–400 μm), it is commonly used in practice, with over half of the surgical procedures employing it [2]. An illustration of the electrosurgical process is shown in Fig. 1.

The electrosurgical control problem can be posed as a boundary control problem characterized by a controllable heat flux, as dictated by the power setting, and cathode position actuation. While the dynamics of heat propagation in the tissue are nontrivial, here we consider as a first approximation a homogeneous substance undergoing a moving phase change from *virgin* to a *denatured* tissue. In particular, we focus on electrosurgical action in a laparoscopic setting, such as laparoscopic liver surgery [3]. This allows casting the ablation control problem as a *Stefan problem* [4], in which the control object is the moving phase-change interface (PCI), and the manipulated variable (i.e., control input) is the heat flux.

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Control of moving boundary heat conduction problems has been treated extensively in the past, albeit mostly in one-phase Stefan problems. Petrus *et al.* have introduced an output feedback enthalpy-based controller design for the one-phase Stefan problem, including a full-state feedback controller and a boundary sensing-based full-state estimator, where the boundary heat flux is the controlled variable [5], [6], [7]. In addition, Petrus *et al.* [8] introduced a state estimation design with online parameter calibration of a single unknown.

In the subsequent work, Koga *et al.* introduced a backstepping controller and observer to control the PCI location of a one-phase Stefan problem using either a variable tip temperature or heat flux, requiring measurements of the PCI location and temperature gradient [9], [10]. Recently, in [11], a two-phase Stefan problem control law was introduced. Chen *et al.* extended the enthalpy-based controller and observer of [6] to account for input hysteresis arising in spray cooling of industrial steel casting processes [12], [13].

The primary control objective here is to maintain a setpoint temperature in the specified vicinity of the probe tip, as demarcated by the PCI location. The secondary objective is to attain the desired PCI position.

Since direct temperature sensing at or near the point of electrosurgical action is often exceedingly hard, temperatures are measured beyond the point of boundary actuation [14], showing the need for a non-collocated feedback control law. Even in denatured tissue, temperature sensors will most likely be damaged due to electrosurgical action, rendering only sensors beyond the PCI location operational. Non-collocated sensing and control has been extensively studied in flexible structures (e.g., [15]), and linear parabolic PDE systems (e.g., [16]).

Along with the aforementioned approaches, Maidi and Courriou [17] addressed mean temperature control of the one-phase Stefan problem with tip heat flux actuation by utilizing the geometric control approach presented by Christofides [18]. This paper uses a similar approach, but extends it to a two-phase Stefan problem with non-collocated pointwise temperature feedback. Here, the proofs rely also on LaSalle's invariance principle for convergence of the observer and reference systems, as well as the PDE maximum principle theorems. To the best of the authors' knowledge, the proposed control law is the first non-collocated pointwise temperature feedback control law applied to the two-phase Stefan problem. In addition, the proposed control law permits an arbitrary number of sensors.

The paper has the following structure. A mathematical

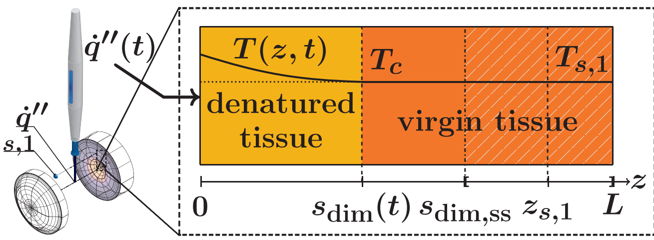

Fig. 1. Exploded view of laparoscopic electrosurgical operation, with the two semi-spheres designating the tissue. Here, s_{dim} denotes the phase-change interface (PCI) location, with $s_{\text{dim, ss}}$ denoting its steady-state, and $z_{s,1}$ denotes the sensor placement, which is assumed to be in the hatched area.

model for the boundary control problem is presented in Sec. II. In Sec. III, we present a non-collocated temperature-based feedback control law of the electrosurgical action along with the proofs of observer convergence and closed-loop stability. In Sec. IV, we demonstrate closed-loop performance through numerical simulation. Conclusions are drawn in Sec. V.

II. The Stefan Problem

We present here a non-dimensional formulation of the two-phase Stefan problem, based on an energy balance at the phase-change interface (PCI) on a 1-D domain [6]:

$$\frac{\partial \theta(\xi,\tau)}{\partial \tau} = \frac{\partial^2 \theta(\xi,\tau)}{\partial \xi^2}, \tag{1}$$

for $\tau > 0$ and $\xi \in (0,1) \setminus \{s(\tau)\}$, subject to

$$-\left.\frac{\partial \theta(\xi,\tau)}{\partial \xi}\right|_{\xi=0} = u(\tau), \qquad \left.\frac{\partial \theta(\xi,\tau)}{\partial \xi}\right|_{\xi=1} = 0, \tag{2}$$

$$\theta(s(\tau),\tau) = \theta_c, \tag{3}$$

with initial conditions

$$\theta(\xi,0) = \theta_0(\xi), \quad s(0) = s_0, \tag{4}$$

and phase-change interface dynamics

$$\frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau} = -\beta \left.\frac{\partial \theta(\xi,\tau)}{\partial \xi}\right|_{\xi=s^-(\tau)}^{\xi=s^+(\tau)}, \tag{5}$$

where $\beta \equiv \frac{c_p(T_{\text{max}}-T_{\text{min}})}{\Delta H_c}$. T_{min} and T_{max} must be defined such that the non-dimensional temperature remains positive, with T_{max} being sufficiently large. The non-dimensional variables are then defined as:

$$\theta(\xi,\tau) \equiv \frac{T(z,t)-T_{\text{min}}}{T_{\text{max}}-T_{\text{min}}},\ \tau \equiv \frac{\alpha t}{L^2},\ \xi \equiv \frac{z}{L},\ s(\tau) \equiv \frac{s_{\text{dim}}(t)}{L}$$

where L is the domain length, $\alpha \equiv k/\rho c_p$ is the molecular thermal diffusivity, k is the thermal conductivity, ρ is the density, and c_p is the isochoric specific heat. Note that in this paper, we assume that α is identical across both phases. In the definition of β, which closely parallels that of a Stefan number, ΔH_c is the enthalpy of denaturation of the tissue. In the above, the phase change (denaturation) occurs at non-dimensional temperature θ_c. Finally, the controlled variable $u(\tau)$, i.e. the non-dimensional heat flux at the boundary, has the following relation with its dimensional counterpart: $u(\tau) = \frac{L}{k(T_{\text{max}}-T_{\text{min}})}\dot q''(t)$.

Note that the second boundary condition prescribed in (2) is not specific to electrosurgery, since insulation at the inner boundary is not guaranteed. However, it does closely match reality provided that the domain length L is chosen sufficiently large, since the effect of electrosurgical action decreases with distance from the site of application [19].

Besides this assumption, we make the following four assumptions:

(A1) The initial conditions satisfy $0 < s_0 < 1$, $\theta_0(\xi) = \theta_c$ for all $0 \le \xi < s_0$, and $\theta_0(\xi) < \theta_c$ and are continuous and non-increasing for all $\xi \ge s_0$, and are piecewise smooth.
(A2) $\theta_0(\xi)$ is continuous on the interval $[0,1]$ and infinitely differentiable inside, except at s_0.
(A3) $\inf u(\tau) \ge 0$ and $\sup u(\tau) < \infty$.
(A4) $\Delta H_c/c_p < T_c$.

Assumption (A1) stipulates that the material is initially under the phase change temperature in the non-denatured, or virgin, tissue, which is physically necessary. It is implicitly assumed that the superheat in the denatured region is negligible at first. Assumption (A3) is physically sound, but not known a priori; it will however hold on the physical system. Assumption (A4) can be verified to hold for most common phase-change processes.

a) Existence and uniqueness: To prove existence and uniqueness of the two-phase Stefan problem above, we may apply the *Picard iteration method* on the non-dimensional system to obtain a unique solution $(\theta(\xi,\tau), s(\tau))$ for known input $u(\tau)$ and with any set of initial conditions $(\theta_0(\xi), s_0)$ that satisfies assumption (A1) [20, Thm. 3, §II.1.1, p. 96]. Thus, well-posedness of the two-phase Stefan problem under consideration is guaranteed.

b) Preliminary properties: By assumptions (A1)–(A3), it can be shown by application of the weak maximum principle [21, Thm. 9, §7.1, p. 390] that $\theta_0(\xi) < \theta_c$ and is non-decreasing for all $s(\tau) \le \xi \le 1$, since $\theta(\xi,\tau)$ attains a minimum value on the parabolic boundary, i.e. $\tau = 0$ or $\xi = s(\tau)$. This in turn implies that $\dot s(\tau) \ge 0$ for all $\tau \ge 0$. Since (1) is parabolic on the subdomains, if (A3) holds $\frac{\partial \theta(\xi,\tau)}{\partial \xi}$ is uniformly bounded (see, e.g., [22, Thm. 11.1, §III.11, p. 211]) by a constant depending on the initial conditions and bounds on u. By (5), this means that the PCI velocity is bounded, i.e.

$$0 \le \dot s_{\text{min}} \le \dot s \le \dot s_{\text{max}} < \infty. \tag{6}$$

Additionally, as a consequence of Poincaré's inequality, there is a bound on $\|\theta\|_2$:

$$\int_0^1 \theta^2(\xi,\tau)\mathrm{d}\xi \le 2\theta_c^2 + 4\int_0^1 \left.\frac{\partial \theta(\xi,\tau)}{\partial \xi}\right|_{\xi=\xi'} \mathrm{d}\xi'.$$

That is, both $\theta(\xi,\tau)$ and $\partial\theta(\xi,\tau)/\partial\xi$ are bounded in the $L_2(0,1)$ norm, and hence $\theta(\xi,\tau)$ is bounded in the Sobolev space $H^1(0,1)$. Similarly, Agmon's inequality ensures that $|\theta(\xi,\tau)|$ is also uniformly bounded.

c) Observer system: Consider an observer of the form:

$$\frac{\partial \hat{\theta}(\xi,\tau)}{\partial \tau} = \frac{\partial^2 \hat{\theta}(\xi,\tau)}{\partial \xi^2} + \mathbf{c}^{\mathsf{T}}(\xi) G_a [\mathbf{y}(\tau) - \hat{\mathbf{y}}(\tau)], \quad (7)$$

where we define $\mathbf{y}(\tau) \equiv \int_0^1 \mathbf{c}(\xi)\theta(\xi,\tau)\,\mathrm{d}\xi$, and the observer gain matrix $G_a \in \mathbb{R}^{n_s \times n_s}$, and $\mathbf{c}(\xi) \equiv \left[c_1(\xi) \cdots c_{n_s}(\xi) \right]^{\mathsf{T}}$ with $c_i(\xi) \equiv \delta(\xi - \xi_{s,i})$ for $i = 1, \ldots, n_s$. Here, $\delta(\cdot)$ denotes the Dirac delta function. These dynamics are subject to the following boundary conditions:

$$-\frac{\partial \hat{\theta}(\xi,\tau)}{\partial \xi}\Big|_{\xi=0} = u(\tau), \quad \frac{\partial \hat{\theta}(\xi,\tau)}{\partial \xi}\Big|_{\xi=1} = 0, \quad (8)$$

$$\hat{\theta}(\hat{s}(\tau),\tau) = \theta_c, \quad \hat{\theta}(\xi,0) = \hat{\theta}_0(\xi). \quad (9)$$

Finally, consider the PCI dynamics given by

$$\frac{\mathrm{d}\hat{s}(\tau)}{\mathrm{d}\tau} = -\beta \left. \frac{\partial \hat{\theta}(\xi,\tau)}{\partial \xi} \right|_{\xi=\hat{s}^-(\tau)}^{\xi=\hat{s}^+(\tau)}. \quad (10)$$

In (7), a Luenberger-type observer feedback term incorporates the non-collocated pointwise temperature feedback as follows, in a similar fashion to [16] as follows.

Define the system's sensed output to be $\mathbf{y}(\tau) \in \mathbb{R}^{n_s}$, where $n_s \geq 1$ denotes the number of (non-collocated) sensors. For each sensor $i \in \mathcal{N}_s \equiv \{1, \ldots, n_s\}$, let $\xi_{s,i} \in (0,1]$ be its location (in-domain, or at the far boundary), at which temperature $\theta_{s,i}(\tau) \equiv \theta(\xi_{s,i},\tau)$ is observed. We assume there be no measurement errors, and that the sensors are farther than the PCI, i.e. $\xi_{s,i} \geq s(\tau)\ \forall\ i \in \mathcal{N}_s, \tau \geq 0$.

d) Reference system: We also define a reference system $\bar{\theta}(\xi,\tau)$ as being of the same form as that of (1)–(5), with the exception that the control input is now $\bar{u}(\tau)$.

III. Non-collocated Feedback Control

Given the systems defined above (nominal, observer, and reference), we can state a number of convergence theorems on the estimation and reference errors.

A. Observer Error Convergence

Theorem 1. *Provided that assumptions (A1)–(A3) hold, the temperature estimation error converges to zero given that observer gain matrix G_a is diagonal and positive semidefinite.*

Proof: Consider the following Lyapunov functional:

$$V_1(\tilde{\theta}) \equiv \frac{1}{2}\int_0^1 \tilde{\theta}^2(\xi,\tau)\,\mathrm{d}\xi - \frac{\theta_c}{\beta}(s(\tau) + \hat{s}(\tau)) + \frac{2\theta_c}{\beta},$$

where the temperature estimation error is defined as $\tilde{\theta}(\xi,\tau) \equiv \theta(\xi,\tau) - \hat{\theta}(\xi,\tau)$.

We assume, without loss of generality, $\hat{s} < s$. Taking the time derivative of the first term of the Lyapunov functional gives:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{2}\int_0^1 \tilde{\theta}^2(\xi,\tau)\mathrm{d}\xi = \frac{\mathrm{d}}{\mathrm{d}\tau}\frac{1}{2}\Bigg(\int_0^{\hat{s}} \tilde{\theta}^2(\xi,\tau)\mathrm{d}\xi + \int_{\hat{s}}^{s} \tilde{\theta}^2(\xi,\tau)\mathrm{d}\xi + \int_s^1 \tilde{\theta}^2(\xi,\tau)\mathrm{d}\xi\Bigg) = -\frac{1}{\beta}\tilde{\theta}(\hat{s},\tau)\frac{\mathrm{d}\hat{s}(\tau)}{\mathrm{d}\tau} + \frac{1}{\beta}\tilde{\theta}(s,\tau)\frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau} - \int_0^1 \left(\frac{\partial \tilde{\theta}(\xi,\tau)}{\partial \xi}\right)^2 \mathrm{d}\xi - \sum_{i \in \mathcal{N}_s} g_{a,i}\tilde{\theta}_{s,i}^2,$$

where $g_{a,i} \equiv G_{a,ii}$, and we have integrated by parts and used the sifting property of the Dirac delta. Differentiating $V_1(\tilde{\theta})$ and substituting the previous expression yields:

$$\frac{\mathrm{d}V_1(\tilde{\theta})}{\mathrm{d}\tau} = -\frac{1}{\beta}\left(\theta(\hat{s},\tau)\frac{\mathrm{d}\hat{s}(\tau)}{\mathrm{d}\tau} + \hat{\theta}(s,\tau)\frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau}\right) - \int_0^1 \left(\frac{\partial \tilde{\theta}(\xi,\tau)}{\partial \xi}\right)^2 \mathrm{d}\xi - \sum_{i \in \mathcal{N}_s} g_{a,i}\tilde{\theta}_{s,i}^2.$$

The first term of this expression is negative, since we can choose a temperature scale such that $\theta, \hat{\theta} > 0$, whereas the velocity of the PCI location has been shown to be nonnegatively bounded from below (see (6)). Finally, ensuring that estimation gain matrix G_a is positive semidefinite ($G_a \geq 0$) will allow us to upper-bound the Lyapunov functional's derivative as follows:

$$\frac{\mathrm{d}V_1(\tilde{\theta})}{\mathrm{d}\tau} \leq -\int_0^1 \left(\frac{\partial \tilde{\theta}(\xi,\tau)}{\partial \xi}\right)^2 \mathrm{d}\xi \leq -\frac{1}{4}\int_0^1 \tilde{\theta}^2(\xi,\tau)\mathrm{d}\xi,$$

where we have applied Poincaré's inequality. By an application of the infinite dimensional invariant set principle [23, Thm. 4.2, §IV.4, p. 168], using an extension of the Rellich–Kondrachov theorem, we can find that $\tilde{\theta}$ asymptotically converges to 0 in $L_2(0,1)$, analogous to the proof of [6, Thm. 1].

B. Observer and Reference Error Convergence

Theorem 2. *Provided that the conditions of Thm. 1 hold, and given $u(\tau) = \bar{u}(\tau)$, both the temperature reference and estimation errors will converge to zero.*

Proof: Let us define $\tilde{\bar{\theta}}(\xi,\tau) \equiv \theta(\xi,\tau) - \bar{\theta}(\xi,\tau)$ and $\tilde{\bar{u}}(\tau) \equiv u(\tau) - \bar{u}(\tau)$.

Consider the following Lyapunov functional candidate:

$$V_2(\tilde{\bar{\theta}},\tilde{\theta}) \equiv \left(\frac{1}{2}\int_0^1 \tilde{\bar{\theta}}^2(\xi,\tau)\mathrm{d}\xi - \frac{\theta_c}{\beta}(s(\tau) + \bar{s}(\tau)) + \frac{2\theta_c}{\beta}\right) + \left(\frac{1}{2}\int_0^1 \tilde{\theta}^2(\xi,\tau)\mathrm{d}\xi - \frac{\theta_c}{\beta}(s(\tau) + \hat{s}(\tau)) + \frac{2\theta_c}{\beta}\right).$$

Taking the time derivative of the first term in the first parentheses, where we assume once again without loss of generality $\bar{s} < s$, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{2}\int_0^1 \tilde{\bar{\theta}}^2(\xi,\tau)\mathrm{d}\xi\right) = \tilde{\bar{\theta}}(0,\tau)\tilde{\bar{u}}(\tau) - \frac{1}{\beta}(\theta(\bar{s},\tau) - \theta_c) \times \frac{\mathrm{d}\bar{s}(\tau)}{\mathrm{d}\tau} - \frac{1}{\beta}(\bar{\theta}(s,\tau) - \theta_c)\frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau} - \int_0^1 \left(\frac{\tilde{\bar{\theta}}(\xi,\tau)}{\partial \xi}\right)^2 \mathrm{d}\xi.$$

Evaluating the rest of the derivative of the Lyapunov functional yields (see Thm. 1):

$$\dot{V}_2 \leq \tilde{\bar{\theta}}(0,\tau)\tilde{\bar{u}}(\tau) - \frac{1}{4}\int_0^1 \left(\tilde{\bar{\theta}}^2(\xi,\tau) + \tilde{\theta}^2(\xi,\tau)\right)\mathrm{d}\xi,$$

where the derivation follows analogously from Thm. 1. Here, both the stipulation that $G_a \geq 0$, as well as Poincaré's inequality have been applied. The use of a nonnegative temperature scale, as well as the existence of a nonnegative underbound to the PCI velocity have also been used (see (6)). By the hypotheses of this proof, $u(\tau) \equiv \bar{u}(\tau)$. Asymptotic convergence in $L_2(0,1)$ can then be proven by application of an invariant set principle argument, analogous to Thm. 1.

C. Non-collocated Output Feedback Controller Design

In the following, we apply Christofides' geometric control theory [18], as was previously applied in [17] on a one-phase Stefan problem with boundary heat flux actuation. We will, however, derive the control law on the estimator system, as opposed to the physical system as was done in [17].

Consider the mean temperature of the estimator over the domain $\xi \in [0, \hat{s}(\tau)]$: $\hat{\theta}_m(\tau) = \int_0^{\hat{s}(\tau)} \hat{\theta}(\xi,\tau)\mathrm{d}\xi$. Taking its derivative with respect to time yields:

$$\frac{\mathrm{d}\hat{\theta}_m(\tau)}{\mathrm{d}\tau} = \left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)} + u(\tau) + (1-\beta\theta_c)\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)}^{\xi=\hat{s}^+(\tau)} + \sum_{i\in\mathcal{N}_s(\hat{s}(\tau))} g_{a,i}\tilde{\theta}_{s,i},$$

where $\mathcal{N}_s(\xi) \equiv \{i \in \mathcal{N}_s : \xi_{s,i} \le \xi\}$.

Since the characteristic index of these dynamics with respect to the manipulated variable $u(\tau)$ is equal to 1, a first-order dynamical behavior between the controlled output $\hat{\theta}_m(\tau)$ and the desired output $\hat{\theta}_m^d(\tau)$ results in [18, Thm. 2.1, §2.4.1, p. 22]:

$$\gamma\frac{\mathrm{d}\hat{\theta}_m(\tau)}{\mathrm{d}\tau} + \hat{\theta}_m(\tau) = \hat{\theta}_m^d(\tau), \tag{11}$$

where γ is the closed-loop tuning parameter. Following this discussion, we find the control law to be:

$$u(\tau) = \frac{1}{\gamma}\left[\hat{\theta}_m^d(\tau) - \hat{\theta}_m(\tau)\right] - \left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)} - (1-\beta\theta_c)\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)}^{\xi=\hat{s}^+(\tau)} - \sum_{i\in\mathcal{N}_s(\hat{s}(\tau))} g_{a,i}\tilde{\theta}_{s,i}. \tag{12}$$

We may now evaluate closed-loop convergence on the estimator system by considering the following error variable: $\hat{\Theta}(\xi,\tau) \equiv \hat{\theta}(\xi,\tau) - \theta_c$.

Theorem 3. *Provided that the conditions of Thm. 2 and assumption (A4) hold, then control law* (12) *will stabilize the closed-loop estimator system.*

Proof: Consider the following Lyapunov functional candidate: $V_3(\hat{\Theta}) \equiv \frac{1}{2}\int_0^1 \hat{\Theta}^2(\xi,\tau)\mathrm{d}\xi$.

Taking the time derivative of this functional yields:

$$\begin{aligned}\frac{\mathrm{d}V_3(\hat{\Theta})}{\mathrm{d}\tau} &= \int_0^1 \hat{\Theta}(\xi,\tau)\frac{\partial\hat{\Theta}(\xi,\tau)}{\partial\tau}\mathrm{d}\xi \\ &= \hat{\Theta}(0,\tau)\left\{\frac{1}{\gamma}\left[\hat{\theta}_m^d(\tau) - \hat{\theta}_m(\tau)\right] - \left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)} - (1-\beta\theta_c)\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)}^{\xi=\hat{s}^+(\tau)}\right\} - \int_0^{\hat{s}(\tau)}\left(\frac{\partial\hat{\Theta}(\xi,\tau)}{\partial\xi}\right)^2\mathrm{d}\xi \\ &+ \sum_{i\in\mathcal{N}_s(\hat{s}(\tau))} g_{a,i}\left[\hat{\Theta}(\xi_{s,i},\tau) - \hat{\Theta}(0,\tau)\right]\tilde{\theta}_{s,i} \\ &+ \sum_{j\in\mathcal{N}_s\setminus\mathcal{N}_s(\hat{s}(\tau))} g_{a,j}\hat{\Theta}(\xi_{s,j},\tau)\tilde{\theta}_{s,j}.\end{aligned}$$

Expanding the second and the third term in braces gives $-\beta\theta_c\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)} - (1-\beta\theta_c)\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^+(\tau)} \le 0$. This inequality follows from assumption (A4) which states that $(1-\beta\theta_c) < 0$, as well as $\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^+(\tau)} \le 0$, and $\left.\frac{\partial\hat{\theta}(\xi,\tau)}{\partial\xi}\right|_{\xi=\hat{s}^-(\tau)} \ge 0$ which both follow from the weak maximum principle [21, Thm. 9, §7.1, p. 390].

Now, turning our attention to the sum containing the term $\hat{\Theta}(\xi_{s,i},\tau) - \hat{\Theta}(0,\tau)$, we wish to show that it is less than or equal to 0. Expanding this term, we obtain $\hat{\Theta}(\xi_{s,i},\tau) - \hat{\Theta}(0,\tau) = \hat{\theta}(\xi_{s,i},\tau) - \hat{\theta}(0,\tau)$.

By the weak maximum principle [21, Thm. 9, §7.1, p. 390], it follows that $\hat{\theta}(0,\tau) \ge \hat{\theta}(\xi_{s,i},\tau)\ \forall\, i \in \mathcal{N}_s$, meaning that $\hat{\Theta}(\xi_{s,i},\tau) - \hat{\Theta}(0,\tau) \le 0$. By the same principle, we have $\hat{\theta}(\hat{s}(\tau),\tau) = \theta_c \ge \hat{\theta}(\xi_{s,j},\tau)$ for all $j \in \mathcal{N}_s \setminus \mathcal{N}_s(\hat{s}(\tau))$.

What is left to show is that $\tilde{\theta}(\xi,\tau) \ge 0$ for $\xi \in (0,1]$, which follows from an argument involving the weak maximum principle and a parabolic version of Hopf's lemma, the details of which are omitted here.

With these results, we may upper-bound the time derivative of the Lyapunov functional as follows:

$$\frac{\mathrm{d}V_3(\hat{\Theta})}{\mathrm{d}\tau} \le \frac{\hat{\Theta}(0,\tau)}{\gamma}\left[\hat{\theta}_m^d(\tau) - \hat{\theta}_m(\tau)\right] - \int_0^1\left(\frac{\partial\hat{\Theta}(\xi,\tau)}{\partial\xi}\right)^2\mathrm{d}\xi.$$

The rest of the proof is equivalent to that of [17, §3.2, pp. 942–3], from which it is found that $\hat{\Theta}(\xi,\tau)$ is exponentially stable in $L_2(0,1)$, and we have $\hat{\Theta}(\xi,\tau) \to 0$ as $\tau \to \infty$ for $\xi \in [0,1]$. With this, we have shown closed-loop convergence for the observer system.

Theorem 4. *Provided that the conditions of Thm. 3 hold, control law* (12) *will stabilize the closed-loop system.*

Proof: The proof is similar to that of Thm. 3. Consider the error variable $\Theta(\xi,\tau) \equiv \theta(\xi,\tau) - \theta_c$. We will consider the following Lyapunov functional candidate: $V_4(\Theta) \equiv \frac{1}{2}\int_0^1 \Theta^2(\xi,\tau)\mathrm{d}\xi$.

Taking the time derivative of this functional yields:

$$\frac{\mathrm{d}V_4(\Theta)}{\mathrm{d}\tau} \leq \frac{\Theta(0,\tau)}{\gamma}[\hat{\theta}_m^d(\tau) - \hat{\theta}_m(\tau)] - \int_0^1 \left(\frac{\partial \Theta(\xi,\tau)}{\partial \xi}\right)^2 \mathrm{d}\xi,$$

where we have omitted steps similar to Thm. 3, and we have applied the properties shown in Thm. 3, as well as a consequence of the weak maximum principle and a parabolic version of Hopf's lemma that yields $\hat{s}(\tau) < s(\tau)$ given the hypotheses of this theorem; the details of this latter argument are omitted here. The rest of the proof is analogous to that of Thm. 3, showing that $\theta(\xi,\tau) \to \theta_c$ as $\tau \to \infty$ for $\xi \in [0,1]$, thus showing closed-loop exponential stability of the physical system in $L_2(0,1)$.

a) Control strategy: Having derived the control law, we now wish to find a way to relate the desired controlled variable $\hat{\theta}_m^d$ to the desired PCI location $s_d(\tau)$. As $\tau \to \infty$, the steady-state mean temperature is related to the steady-state PCI location as $\hat{\theta}_{m,\mathrm{ss}} = \theta_c \hat{s}_{\mathrm{ss}}$; here, the subscript 'ss' denotes the steady-state. This result is readily shown by considering the deviation variable introduced in Thm. 4 [17]. Indeed, we have $\theta_m(\tau) = \int_0^{\hat{s}(\tau)} \hat{\Theta}(\xi,\tau)\mathrm{d}\xi + \theta_c \hat{s}(\tau)$. As $\tau \to \infty$, we have $\hat{\Theta}(\xi,\tau) \to 0$, which gives $\hat{\theta}_{m,\mathrm{ss}} = \int_0^{\hat{s}(\tau)} 0 \mathrm{d}\xi + \theta_c \hat{s}_{\mathrm{ss}}$. A similar argument holds for Θ and $s_d(\tau)$.

Thus, if we take $\hat{\theta}_m^d(\tau) = \theta_c \hat{s}_d(\tau)$, where we set $\hat{s}_d(\tau) = s_d(\tau)$, we may implement the control law as:

$$u(\tau) = \frac{1}{\gamma}\left[\theta_c s_d(\tau) - \hat{\theta}_m(\tau)\right] - \left.\frac{\partial \hat{\theta}(\xi,\tau)}{\partial \xi}\right|_{\xi=\hat{s}^-(\tau)} - (1-\beta\theta_c)\left.\frac{\partial \hat{\theta}(\xi,\tau)}{\partial \xi}\right|_{\xi=\hat{s}^-(\tau)}^{\xi=\hat{s}^+(\tau)} - \sum_{i \in \mathcal{N}_s(\hat{s}(\tau))} g_{a,i}\tilde{\theta}_{s,i}. \tag{13}$$

Using this control law, all parameters are obtained from the known observer system (7)–(10). We now formulate a conjecture to support the discussion given above.

Conjecture 1. *Suppose that* (θ, s), $(\bar{\theta}, \bar{s})$, *and* $(\hat{\theta}, \hat{s})$ *all satisfy the conditions of Thm. 4. Then, under the control law* (13)*, the PCI reference and estimation errors converge to 0.*

IV. Numerical Simulation

The performance of the non-collocated feedback controller is considered through numerical simulation of the controlled ablation of a $L = 5$ mm slab of porcine tissue, with thermodynamic properties being as follows: $T_{\max} = 423.15$ K, $T_{\min} = 273.15$ K, $T_c = 355.15$ K, $c_p = 4$ kJ/kg/K, ΔH_c kJ/kg, $\rho = 700$ kg/m^3, and $k = 0.5934$ W/kg/K (see [24], as well as own experiments). This simulation is based on a variable time step finite difference scheme presented in [25], with simultaneous simulation of the physical and observer systems. The initial values used are $s(0) = 0.06$, $\hat{s}(0) = 0.04$, and $s_d = 0.4$. The initial temperature profile is taken to be $\theta_c + 10^{-6}$ in upstream of the respective PCI locations, and is taken to be $0.05\theta_c$ (7.5 K) lower for the virgin tissue, with the estimator system having an initial

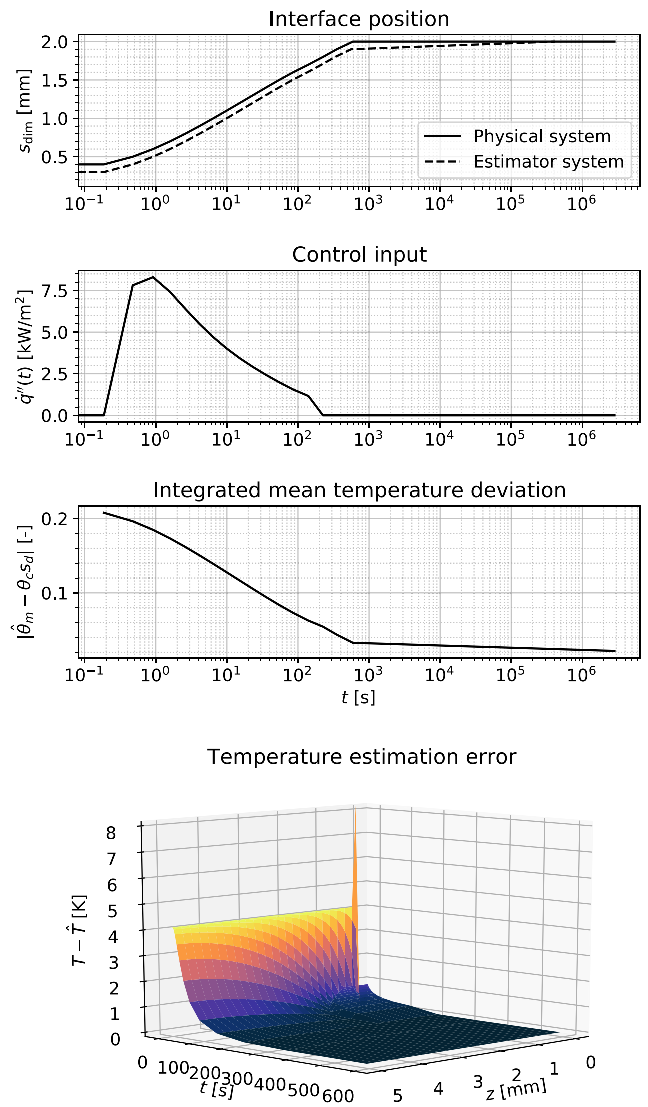

Fig. 2. Closed-loop interface response of system (1)–(5) given setpoint $s_d = 0.4$ and control law (13).

virgin temperature lowered by that same amount. Tuning parameters are taken as $\gamma = 50$ and $G_a = 0.1$, for a single sensor at $\xi_{s,1} = 0.5$.

Conjecture 1 can be demonstrated to hold in Fig. 2, which features physically consistent control input [2], and slow convergence in line with the desire to achieve safe and steady control in electrosurgical settings. While assumption (A3) is clearly satisfied in this simulation, it is possible to find cases where this is not the case, for example by increasing the magnitude of γ and G_a; investigating and mitigating these cases will be the subject of our future work.

This example clearly demonstrates that it is possible to utilize non-collocated sensing in the virgin phase to control the PCI location through actuation on the boundary of the denatured phase. In practice, this would allow for sensor placement that is unhindered by interference due to the electrosurgical probe, while the sensors themselves would not need to be able to detect temperatures much higher than the phase change temperature.

V. Conclusion

In this work, we have considered the control of an electrosurgical procedure. We have derived directly from the governing equations of the two-phase Stefan problem a non-collocated feedback control law, and have proven exponential stability of the closed loop. Our control law is based on non-collocated pointwise temperature sensing and control feedback through an observer system, demonstrating for the first time a physically viable electrosurgical control setup. We have proven convergence of the estimator system, and posed a conjecture on PCI location convergence. Both the estimator temperature error convergence and the interface position convergence have been shown through simulation, while the latter still requires a formal proof.

Going forward, we will study approaches to ensure system stability under input saturation constraining the input to be nonnegative; one possible approach is to implement adaptive gain changes based on the sensor readings. In parallel, we plan to study output error injection due to sensing errors, and derive guarantees on system convergence for known output error characteristics.

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